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An Application of a Theorem of De Bruijn, Tengbergen, and Kruyswijk

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ABSTRACT

Let $\{a_i\}$ be an increasing sequence of positive integers containing no three distinct elements a_i, a_j, a_k, for which the lowest common multiple of a_i, a_j is equal to a_k. By using the theorem mentioned in the title, we prove that for sufficiently large n,

$$\sum_{a_i \leqslant n} \frac{1}{a_i} < (2\sqrt{2/\pi} + \epsilon) \frac{\log n}{\sqrt{\log \log n}}.$$

Erdös, Sárkösy, and Szemerédi [1] have proved the existence of an absolute constant C such that, if $\{a_i\}$ is an increasing sequence of positive integers for which

$$\sum_{a_i \leqslant n} \frac{1}{a_i} > C \frac{\log n}{\sqrt{\log \log n}},$$

then there are distinct elements a_i, a_j, a_k such that[1] $[a_i, a_j] = a_k$. The proof depends on a striking combinatorial result due to Kleitman [2]. Kleitman's result is the square-free case of the lemma below, and is based on an old theorem of Sperner [3]. In the form relevant to this paper, Sperner's theorem asserts that a set of divisors of the square-free number

$$m = \prod_{i=1}^{n} p_i,$$

no one divisor dividing another, can contain at most $\binom{n}{[n/2]}$ elements. The generalization of this theorem to any m, not necessarily square-free,

[1] $[a, b]$ denotes the least common multiple of a and b.

is included in a result of De Bruijn, Tengbergen, and Kruyswijk [4], which we now state.

We define the *degree* $\Omega(m)$ of $m = \prod_i p_i^{\alpha_i}$ to be $\Omega(m) = \sum_i \alpha_i$, and denote by $s(m)$ the number of divisors of m of degree $[\frac{1}{2}\Omega(m)]$. We say further that a set $d_1, \ldots, d_h$ of divisors of m which satisfy the conditions

$$\frac{d_{i+1}}{d_i} \text{ is a prime} \qquad (i = 0, \ldots, h-1), \tag{1}$$

$$\Omega(d_1) + \Omega(d_h) = \Omega(m), \tag{2}$$

form a *symmetric chain*. The authors of [4] proved in a very simple way that the divisors of m can be placed in $s(m)$ disjoint-symmetric chains.[2] It follows immediately that a set of divisors of m, none dividing another, can contain at most one element from each chain and hence at most $s(m)$ elements.

In this note we show how this result can be applied to obtain a numerical upper bound for the best possible value of C. We first combine it with Kleitman's elegant method to obtain the following combinatorial result.

LEMMA. *Let $A = \{a_1, \ldots, a_r\}$ be a set of divisors of m such that for all i, j, k, $[a_i, a_j] \neq a_k$. Then*

$$r \leqslant \tau(m) \min_{\substack{m=uv \\ (u,v)=1}} \left\{ \frac{s(u)}{\tau(u)} + \frac{s(v)}{\tau(v)} \right\}$$

where τ denotes the divisor function.

PROOF: Consider a particular choice of u, v and place the divisors b_i of u in $s(u)$ disjoint-symmetric chains. Let us denote the divisors of v by c_i. For each c_i consider the first b in each symmetric chain for which $bc_i \in A$, and denote the set of all elements of A arising from c_i in this way by S_i. Then, since there are $\tau(v)$ choices of c_i, $\bigcup_i S_i$ contains at most $s(u)\tau(v)$ elements of A. Let $A' = A - \bigcup_i S_i$, and suppose, if possible, that there exist b_1, c_1, c_2 such that $b_1c_1 \in A'$, $b_1c_2 \in A'$, $c_1 \mid c_2$. Then there exists b_2, in the same chain as b_1, for which $b_2 \mid b_1$ and $b_2c_2 \in A - A'$. We then have $[b_1c_1, b_2c_2] = b_1c_2$, which contradicts the hypothesis of the lemma. It follows by the theorem in [4] that for a fixed b_i there are at most $s(v)$ divisors c_j of v such that $b_ic_j \in A'$. Thus A' contains at most $\tau(u)s(v)$ elements and hence there are at most $s(u)\tau(v) + s(v)\tau(u)$ elements in A.

[2] This theorem is related to a result of Dilworth on partially ordered sets (*Annals of Math.* **51**, 161–166).

It is proved in [5] that

$$s(m) \leqslant \frac{\tau(m)}{2^{\Omega(m)}} \binom{\Omega(m)}{[\frac{1}{2}\Omega(m)]}.$$

On applying Stirling's formula we obtain, for any given $\epsilon > 0$, an integer $N = N(\epsilon)$ such that

$$s(m) \leqslant \left(\sqrt{\frac{2}{\pi}} + \epsilon\right) \frac{\tau(m)}{\sqrt{\Omega(m)}}$$

provided $\Omega(m) > N(\epsilon)$. We thus have

COROLLARY. *If m can be expressed as $m = m_1 m_2$ where $(m_1, m_2) = 1$ and $\Omega(m_i) \geqslant (\frac{1}{2} - \epsilon)\Omega(m)$ $(i = 1, 2)$, then if $\Omega(m) > N(\epsilon)$,*

$$r \leqslant \left(\frac{4}{\sqrt{\pi}} + \epsilon\right) \frac{\tau(m)}{\sqrt{\Omega(m)}}.$$

We are now in a position to prove our main result.

THEOREM. *Let $\{a_i\}$ be an increasing sequence of positive integers which does not contain distinct elements a_i, a_j, a_k such that $[a_i, a_j] = a_k$. Then for $n > n(\epsilon)$,*

$$\sum_{a_i \leqslant n} \frac{1}{a_i} \leqslant \left(2\sqrt{\frac{2}{\pi}} + \epsilon\right) \frac{\log n}{\sqrt{\log \log n}}.$$

PROOF: Let n be a sufficiently large positive integer. We have

$$n \sum_{a_i \leqslant n} \frac{1}{a_i} = \sum_{a_i \leqslant n} \left[\frac{n}{a_i}\right] + O(n) = \sum_{m \leqslant n} r(m) + O(n)$$

where $r(m)$ denotes the number of a_i dividing m. Thus by the corollary to the lemma, with $r = r(m)$,

$$\sum_{a_i \leqslant n} \frac{1}{a_i} \leqslant \left(\frac{4}{\sqrt{\pi}} + \epsilon\right) \frac{1}{n} \sum_{m \leqslant n}{}' \frac{\tau(m)}{\sqrt{\Omega(m)}} + \frac{1}{n} \sum_{m \leqslant n}{}'' \tau(m) + O(1),$$

where $\sum'$ extends over all $m \leqslant n$ satisfying the conditions of the above corollary, and $\sum''$ extends over all other m.

Define the iterated logarithmic function $l_i(n)$ by

$$l_1(n) = \log n, \qquad l_{i+1}(n) = l(l_i(n)) \qquad (i = 1, 2, \ldots).$$

Using the fact that

$$\sum{}' 1 = o(n) \qquad \text{as} \quad n \to \infty,$$

where summation is over all $m \leqslant n$ such that

$$| \Omega(m) - l_2(n) | > (l_2(n))^{\frac{1}{2}+\delta},$$

it is proved in [5] by an elementary argument that

$$\sum_{m \leqslant n} \frac{\tau(m)}{\sqrt{\Omega(m)}} \sim \frac{n l_1(n)}{\sqrt{2 l_2(n)}}, \qquad n \to \infty.$$

Hence it follows that

$$\sum_{a_i \leqslant n} \frac{1}{a_i} \leqslant \left(2\sqrt{\frac{2}{\pi}} + \epsilon\right) \frac{l_1(n)}{\sqrt{l_2(n)}} + \frac{1}{n} \sum_{m \leqslant n}{}'' \tau(m). \tag{3}$$

Further, since

$$\sum_{\Omega(m) \leqslant l_2(n)} \tau(m) \leqslant \sum_{m \leqslant n} 2^{l_2(n)} = o\left(\frac{n l_1(n)}{\sqrt{l_2(n)}}\right),$$

we need consider in $\sum''$ only those m for which $\Omega(m) > l_2(n)$. Then $\sum''$ extends over all $m \leqslant n$ with $\Omega(m) > l_2(n)$, which cannot be written in the form $m = m_1 m_2$, where $(m_1, m_2) = 1$ and

$$\Omega(m_i) \geqslant (\tfrac{1}{2} - \epsilon)\Omega(m).$$

Now clearly all such m can be expressed in the form $m = p^\alpha u$ where

$$\alpha = \alpha(n) = \left[\frac{2 l_3(n)}{l_1(2)}\right] + 1$$

(and where p may or may not divide u). Thus, since

$$\tau(p^\alpha u) \leqslant \tau(p^\alpha)\tau(u),$$

we have finally

$$\begin{aligned} \sum_{m \leqslant n}{}'' \tau(m) &= O\left\{ \sum_{\substack{p,u \\ p^\alpha u \leqslant n}} \tau(p^\alpha)\, \tau(u) \right\} \\ &= O\left\{ l_3(n) \sum_{\substack{p,u \\ p^\alpha u \leqslant n}} \tau(u) \right\} \end{aligned}$$

$$= O\left\{l_3(n) \sum_{p\leqslant n^{1/\alpha}} \frac{n}{p^\alpha}\, l_1\left(\frac{n}{p^\alpha}\right)\right\}$$

$$= O\left\{nl_1(n)\, l_3(n) \sum_{m=2}^{\infty} \frac{1}{m^\alpha}\right\}$$

$$= O\left\{nl_1(n)\, l_3(n) \frac{1}{2^\alpha}\right\}$$

$$= O\left\{\frac{nl_1(n)}{\sqrt{l_2(n)}}\right\}.$$

The theorem now follows from (3).

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